

Asymptotic behaviour of a random walk killed on a finite set

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Abstract

We study asymptotic behavior, for large time n , of the transition probability of a two-dimensional random walk killed when entering into a non-empty finite subset A . We show that it behaves like $4\tilde{u}_A(x)\tilde{u}_{-A}(-y)(\lg n)^{-2}p^n(y-x)$ for large n , uniformly in the parabolic regime $|x| \vee |y| = O(\sqrt{n})$, where $p^n(y-x)$ is the transition kernel of the random walk (without killing) and \tilde{u}_A is the unique harmonic function in the ‘exterior of A ’ satisfying the boundary condition $\tilde{u}_A(x) \sim \lg |x|$ at infinity.

1 Introduction and main results

Let $S_n = S_0 + X_1 + \cdots + X_n$, $n = 1, 2, \dots$ be a random walk on the d -dimensional square lattice \mathbb{Z}^d , $d \geq 2$, defined on some probability space (Ω, \mathcal{F}, P) . Here the increments X_j are i.i.d. random variables taking values in \mathbb{Z}^d ; the initial state S_0 may be any random variable specified according to the occasion. As usual the conditional law given $S_0 = x$ of the walk (S_n) is denoted by P_x and the expectation under it by E_x . Let A be a non-empty finite subset of \mathbb{Z}^d . We are concerned with the asymptotic behavior of

$$p_A^n(x, y) = P_x[S_k \notin A \text{ for } k = 1, \dots, n \text{ and } S_n = y],$$

the transition probability matrix of the walk killed on hitting A . In the classical paper [2] H. Kesten proved among others that if $d = 2$ and the walk is recurrent and (temporally) aperiodic, then for each $x \in \mathbb{Z}^2$ and $y \in \mathbb{Z}^2 \setminus A$, as $n \rightarrow \infty$

$$p_A^n(x, y) = u_A(x)u_{-A}(-y) \sum_{\xi \in A} q_A(\xi, n)(1 + o(1)). \quad (1)$$

Here $q_A(\xi, n)$ is the P_ξ -probability of the walk returning to A at time n for the first time and u_A is a unique harmonic function for the killed walk (i.e., $E_x[u_A(S_1); S_1 \notin A] = u_A(x)$ for all $x \in \mathbb{Z}^2$) that satisfies $\sum_{\xi \in A} u_A(\xi) = 1$ (the formulation is slightly modified from [2] in which the function $P_x[S_k \notin A \text{ for } k = 1, \dots, n-1 \text{ and } S_n = y]$ is considered instead of p_A^n). In this paper we improve the above asymptotic formula under existence of the finite second moments, so that it is valid uniformly within the parabolic region $|x| \vee |y| < M\sqrt{n}$ for each $M \geq 1$. Our approach is different from that of [2] and does not depend on the

result of [2]. The proof in [2] is done by compactness arguments, namely by showing the convergence of the ratio $p_A^n(x, y) / \sum_{\xi \in A} q_A(\xi, n)$ as $n \rightarrow \infty$ along subsequences and identifying the limit. In our approach, suggested by intuitive idea of how random walk paths must behave to contribute to the transition probability, we directly compute the limit by using an estimate of the probability of the walk making a large excursion without hitting A . For the estimation of such a probability the potential function of the random walk with its fundamental properties established by Spitzer [6] plays a substantial role as in [2].

For one dimensional recurrent and aperiodic walk Kesten [2] proved that if $EX^2 = \infty$, then (1) is valid, and if $EX^2 = \sigma^2 < \infty$ and $A = \{0\}$, then for each $x, y \neq 0$

$$p_A^n(x, y) = [a(x)a(-y) + \sigma^{-4}xy]q_{\{0\}}(0, n)(1 + o(1)),$$

where $a(x)$ is a potential function of the walk. The problem of extending the latter result to a general A consisting of more than one points is studied in a separate paper [13]. In the higher dimensional case $d \geq 3$, where Kesten [2] also gives a definite result similar to (1) in a quite general framework, the uniform estimate is readily obtained if the existence of the second moments is assumed as we shall briefly mention at the end of this section.

The corresponding problem for two-dimensional Brownian motion is dealt with by the present author [12]. Although the strategy of the proof is the same for the both processes, the Brownian case is basically simpler and for it we can obtain quite detailed estimates valid uniformly beyond parabolic region, whereas adaptation of the proof to the random walk case, requiring us to manage the overshoots to obtain relevant estimates, is nontrivial even though the space-time parameters are restricted to a parabolic region.

Throughout this paper we suppose that the random walk S_n is irreducible (i.e., for every $x \in \mathbb{Z}^d$, $P_0[S_n = x] > 0$ for some $n > 0$) and that

$$EX = 0 \quad \text{and} \quad E|X|^2 < \infty. \quad (2)$$

Here X is a random variable having the same law as X_1 and $|\cdot|$ the usual Euclidian norm. Let $p^n(x) = P_0[S_n = x]$ so that $p^0(x) = \delta(0, x)$ (Kronecker's delta kernel) and

$$P_x[S_n = y] = p^n(y - x).$$

For a non-empty set $B \subset \mathbb{Z}^d$, σ_B (resp. τ_B) denotes the first time when S_n enters into (resp. exits from) B :

$$\sigma_B = \inf\{n \geq 1 : S_n \in B\}, \quad \tau_B = \inf\{n \geq 1 : S_n \notin B\}.$$

(We shall sometimes write $\sigma(B)$ for σ_B for typographical reason and similarly for $\tau(B)$.) By means of σ_B the transition probability of S_n killed on entering B is written as

$$p_B^n(x, y) = P_x[S_n = y, \sigma_B > n], \quad n = 0, 1, 2, \dots;$$

in particular

$$p_B^0(x, y) = \delta(x, y) \quad \text{and} \quad p_B^1(x, y) = p(y - x)(1 - \chi_B(y))$$

for all x, y , where $\chi_B(y) = 1$ or 0 according as $y \in B$ or $y \notin B$. Note that $p_B^n(x, y) = 0$ whenever $y \in B, n \geq 1$.

Let $A \subset \mathbb{Z}^d$ be a non-empty finite set such that the walk killed on A is irreducible, so that

$$\forall x \notin A, \forall y \notin A, \exists n \geq 1, \quad p_A^n(x, y) > 0, \quad (3)$$

which is supposed throughout the sequel. This imposes no essential restriction (see Remark 1(a) after Theorem 1).

Let $d = 2$. Denote the Green function associated with p_A^n by $g_A(x, y)$:

$$g_A(x, y) = \sum_{n=0}^{\infty} p_A^n(x, y), \quad x, y \in \mathbb{Z}^2$$

and put

$$u_A(x) = \lim_{|y| \rightarrow \infty} g_A(x, y), \quad x \in \mathbb{Z}^2. \quad (4)$$

(Cf. [7, Theorem 14.3] for the existence of the limit.) $u_A(x)$ may be interpreted as the expected number of visits to x made by the dual (or time-reversed) random walk ‘starting at infinity’ up to (inclusively) the time of first entrance into A . Let Q be the covariance matrix of X , namely the 2×2 matrix whose quadratic form equals $E(X \cdot \theta)^2$, $\theta \in \mathbb{R}^2$, and put

$$\kappa = \pi \sqrt{\det Q}.$$

As we shall see in Section 2.3 it holds that as $|x| \rightarrow \infty$

$$\kappa u_A(x) \sim \log |x|. \quad (5)$$

Note that for $x \in A$, $u_A(x) > 0$ if and only if $p(y - x) > 0$ for some $y \notin A$ (under condition (3)).

Theorem 1. *Let $d = 2$ and A be a finite subset of \mathbb{Z}^2 that is non-empty and satisfies (3). Then, for each $M \geq 1$, uniformly for $x \in \mathbb{Z}^2$ and $y \in \mathbb{Z}^2 \setminus A$ subject to the constraint $|x| \vee |y| < M\sqrt{n}$, as $n \rightarrow \infty$*

$$p_A^n(x, y) = \frac{4\kappa^2 u_A(x) u_{-A}(-y)}{(\lg n)^2} p^n(y - x) (1 + o(1)). \quad (6)$$

REMARK 1. (a) If condition (3) is not assumed, it may possibly occur that $u_A(x) u_{-A}(-y) = 0$ and $p_A^n(x, y) > 0$ for some $x \in \mathbb{Z}^2$ and $y \notin A$ and for infinitely many n ; thus (6) is not always true. However, if $u_A(x) u_{-A}(-y) = 0$, then the random walk paths that connect x and y and avoid $A \setminus \{x, y\}$ are all confined in any disc that contains A , hence $p_A^n(x, y)$ approaches zero exponentially fast and we may consider only the case $u_A(x) u_{-A}(-y) > 0$, or, what amounts to substantially the same thing, augment A by adding all x with $u_A(x) u_{-A}(-x) = 0$ so that (3) is satisfied by the resulting set.

(b) Under the stronger moment condition $E[|X|^{2+\delta}] < \infty$ (for some $\delta > 0$) the error term $o(1)$ can be replaced by $o([\lg \lg n] / \lg n)$ in (6). Although we shall not give full proof of it, some estimates needed for it will be provided.

(c) Our definition of g_A is not standard. In [7] and [5] the Green function for the walk killed on A is given by $G_A(x, y) = \sum_{n=0}^{\infty} P_x[S_k \notin A (k = 0, \dots, n), S_n = y]$ so that $G_A(x, y) = 0$ if either $x \in A$ or $y \in A$, but still $g_A(x, y) = G_A(x, y)$ whenever $x \notin A$. If the hitting distribution $H_A(x, y)$ is defined to be equal to $P_x[S_{\sigma(A)} = y]$ if $x \notin A$ and $\delta(x, y)$ if

$x \in A$, then its dual $\widehat{H}_A(x, y)$ say, i.e., the corresponding distribution for the dual process $\widehat{S}_n := \widehat{S}_0 - X_1 - \cdots - X_n$, being equal to $H_{-A}(-x, -y)$, we have

$$g_A(x, y) = G_A(x, y) + H_{-A}(-y, -x).$$

By means of this g_A we shall have a neat expression of $u_A(x)$ (see Lemma 2.9).

Taking limit in $g_A(x, y) = \delta(x, y) + \sum_{z \notin A} p(z-x)g_A(z, y)$ we deduce that u_A is harmonic for the walk killed on A in the sense that

$$u_A(x) = E_x[u_A(S_1); S_1 \notin A] \quad \text{for all } x \in \mathbb{Z}^2. \quad (7)$$

For $\xi \in A$, in particular, we have $u_{-A}(-\xi) = \sum_{y \notin A} u_{-A}(-y)p(\xi - y)$. Keeping this in mind we substitute the expression given in Theorem 1 for $p_A^{n-1}(x, y)$ in the identity

$$P_x[\sigma_A = n, S_n = \xi] = \sum_{y \notin A} p_A^{n-1}(x, y)p(\xi - y),$$

and notice that in view of a local limit theorem $p^{n-1}(y-x)$ is well approximated by $p^n(\xi-x)$ for any sufficiently large n and any y with $p^{n-1}(y-x)p(\xi-y) > 0$ uniformly under the constraints $|y| = o(\sqrt{n})$ and $|x| < M\sqrt{n}$, which leads to the following

Corollary 1. *Let $d = 2$ and A be a finite subset of \mathbb{Z}^2 that is non-empty and satisfies (3). Then, for each $M > 1$ and for $\xi \in A$, uniformly for $x \in \mathbb{Z}^2$ with $|x| < M\sqrt{n}$, as $n \rightarrow \infty$*

$$P_x[\sigma_A = n, S_n = \xi] = \frac{4\kappa^2 u_A(x) u_{-A}(-\xi)}{(\lg n)^2} p^n(\xi - x)(1 + o(1)). \quad (8)$$

The function u_A restricted on A may be regarded as the hitting distribution of A for the dual walk started at infinity (since the walk visits A exactly once in the interval $[0, \sigma_A]$; cf. also Remark 1 (c)). It is noted that this fact may be expressed as

$$u_{-A}(-\xi) = \lim_{|y| \rightarrow \infty} P_y[S_{\sigma_A} = \xi], \quad \xi \in A.$$

It follows that $\sum_{\xi \in A} u_A(\xi) = 1$ and we deduce from (8) that *if the walk is aperiodic, then*

$$P_x[\sigma_A = n] = \frac{4\kappa^2 u_A(x)}{(\lg n)^2} p^n(-x)(1 + o(1))$$

and

$$\sum_{\xi \in A} P_\xi[\sigma_A = n] = \sum_{\xi \in A} q_A(\xi, n) = \frac{4\kappa^2}{(\lg n)^2} p^n(0)(1 + o(1));$$

in particular the formula (6) conforms to (1) for each x, y fixed.

The following proposition, crucial in our proof of Theorem 1, may explain why the function $u_A(x)$ comes into formula (6). Let $U(R)$, $R > 0$ denote the disc of radius R :

$$U(R) = \{x \in \mathbb{Z}^2 : |x| < R\}.$$

Proposition 1. *Uniformly for $x \in U(R)$, as $R \rightarrow \infty$*

$$P_x[\tau_{U(R)} < \sigma_A] = \frac{\kappa u_A(x)}{\lg R} (1 + o(1));$$

and if $E[X^2 \lg |X|] < \infty$, then the error term $o(1)$ can be replaced by $o(1/\lg R)$.

For the description of the strategy of the proof of Theorem 1 the readers are referred to [12]: in the latter half of its first section the skeleton of proof to the corresponding result for Brownian motion is given, by which the role of Proposition 1 may be fully understood.

Proof of Proposition 1 is given at the end of Section 2, where some known results and immediate consequences of them are stated, overshoots estimates are discussed, and the function u_A is defined in a way apparently different from (4). Theorem 1 is proved in Section 3 after two propositions are shown.

We conclude this section by making a short mention of the higher dimensional case. Let $d \geq 3$ and define

$$u_A(x) = P_x[\sigma_A = \infty].$$

As an analogue of Theorem 1 we can then verify that uniformly for $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d \setminus A$ satisfying $|x| \vee |y| < M\sqrt{n}$, as $n \rightarrow \infty$

$$p_A^n(x, y) = u_A(x)u_{-A}(-y)p^n(y - x)(1 + o(1)). \quad (9)$$

In view of the bound

$$\sum_{k=0}^{\infty} p^k(x) = o(1) \quad (|x| \rightarrow \infty)$$

as well as the trivial facts: $P_x[\sigma_{\{0\}} = n] \leq p^n(-x)$; $u_A(x) \rightarrow 1$ ($|x| \rightarrow \infty$); and

$$P_x[\sigma_{U(R)} < \sigma_A] \rightarrow u_A(x) \quad (R \rightarrow \infty)$$

the proof of Theorem 1 is easily adapted for verification of the formula (9) with much simplification.

2 Overshoot estimates and proof of Proposition 1

2.1 Preliminary results

Here we collect known results or its refined version that are used in the succeeding sections. Let $d = 2$. Put

$$a^\dagger(x) = \delta(x, 0) + a(x),$$

where $\delta(x, y) = 0$ or 1 according as $x \neq y$ or $x = y$. We recall that for all x, y

$$a(x) := \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)] \geq 0,$$

$$E_x[a(S_1 - y)] = a^\dagger(x - y), \quad (10)$$

and

$$g_{\{0\}}(x, y) = \delta(x, 0) + a(x) + a(-y) - a(x - y); \quad (11)$$

and that as $|x| \rightarrow \infty$

$$a(x) = \frac{1}{\pi\sqrt{\det Q}} \lg |x| + o(\lg x), \quad (12)$$

$$a(x + y) - a(x) \rightarrow 0 \quad \text{for each } y \quad (13)$$

(cf. [5, Theorem 4.4.6] for (12) and [7, Propositions 11.6 and 12.2] for (11) and (13)). By (11) one can readily verify that if $\xi \in A$, the process $M_n := a(S_{n \wedge \sigma_A} - \xi)$ is a martingale relative to the filtration of the stopped process $S_{n \wedge \sigma_A}$ under P_x for $x \neq \xi$.

We write x^2 for $|x|^2$. In Lemmas 2.1 and 2.2 below ν denotes the period of the walk S , i.e., ν is the g.c.d. of $\{n : p^n(0) > 0\}$.

Lemma 2.1. *Let $d = 2$. Then, (i) uniformly for $x \in \mathbb{Z}^2$ with $p^n(-x) > 0$, as $n \rightarrow \infty$*

$$P_x[\sigma_{\{0\}} = n] = \frac{2\nu\kappa a^\dagger(x)}{n(\lg n)^2}(1 + o(1)) + O\left(\frac{x^2 \vee 1}{n^2 \lg n}\right), \quad (14)$$

and (ii) as $n \wedge |x| \rightarrow \infty$

$$P_x[\sigma_{\{0\}} = n] = \frac{4\kappa \lg |x|}{(\lg n)^2} p^n(-x) + o\left(\frac{1}{n \lg n} \left(1 \wedge \frac{\sqrt{n}}{|x|}\right)\right). \quad (15)$$

Proof. These are proved in [11]: the first formula is a reduced version of Theorem 1.3, the second is the first half of Theorem 1.4 (see also the next lemma). \square

The next result taken from [10, Corollary 6] is a version of local central limit theorems for the random walk on \mathbb{Z}^d , $d \geq 1$. (Corollary 6 of [10], stated for aperiodic case, is adapted to general case in an obvious way.) Put

$$p_t^{(d)}(x) = \frac{1}{(2\pi t)^{d/2}} e^{-x^2/2t}$$

and

$$\sigma = (\det Q)^{1/2d}, \quad \tilde{x} = \sigma \sqrt{Q^{-1}} x \quad \text{and} \quad \|x\| = |\tilde{x}| = \sigma \sqrt{x \cdot Q^{-1} x}.$$

(The factor σ is put to make $\|x\|$ agree with $|x|$ when Q is isotropic.)

Lemma 2.2. *Suppose $E[|X|^{2+\delta}] < \infty$ for a constant $0 \leq \delta < 1$. Then for $n \geq 1$ and x with $p^n(x) > 0$,*

$$p^n(x) = \nu p_{\sigma^2 n}^{(d)}(\tilde{x}) + \frac{1}{n^{d/2}} \times o\left(\frac{n}{(\sqrt{n} \vee |x|)^{2+\delta}}\right),$$

as $n + |x| \rightarrow \infty$ (for each $d = 1, 2, \dots$).

Lemma 2.3. *Let $d = 2$ and $n \geq 2$. Then as $|x| \wedge n \rightarrow \infty$*

$$P_x[\sigma_{\{0\}} < n] = \frac{1}{\lg n} \int_{\tilde{x}^2/n}^{\infty} e^{-u/2\sigma^2} \frac{du}{u} + o\left(1 \wedge \frac{\sqrt{n}}{|x| \lg n}\right); \quad (16)$$

in particular $P_x[\sigma_{\{0\}} < n] \rightarrow 0$ if $\liminf[(\lg x^2)/\lg n] \geq 1$.

Proof. For simplicity suppose that the walk is aperiodic. First suppose $|x| > \sqrt{n}/\lg n$. Using (15) for $\lg n < k < n$ and $P_x[\sigma_{\{0\}} = k] \leq p^k(-x)$ for $k \leq \lg n$ we compute the sum of $P_x[\sigma_{\{0\}} = k]$ over $k < n$. By Lemma 2.2 we have $2\kappa p^k(-x) = k^{-1} e^{-\lambda \tilde{x}^2/k} + o(1/(k \vee x^2))$,

where $\lambda = 1/2\sigma^2$, and with the sum $\sum_{k=\lg n}^n (\lg k)^{-2} e^{-\lambda \tilde{x}^2/k} / k$ computed in a usual manner we observe

$$\begin{aligned} P_x[\sigma_{\{0\}} < n] &= o\left(\frac{\lg n}{x^2}\right) + 2\lg|x| \int_{\tilde{x}^2/n}^{\tilde{x}^2/\lg n} \frac{e^{-\lambda u} u^{-1} du}{[\lg(\tilde{x}^2/u)]^2} (1 + o(1)) \\ &\quad + \sum_{\lg n < k < n} \left(\frac{\lg|x|}{(\lg k)^2 (k \vee x^2)} + \frac{1}{|x| \sqrt{k} \lg k} \right) \times o(1) \\ &= \frac{2\lg|x|}{(\lg n)^2} \int_{\tilde{x}^2/n}^{\infty} e^{-\lambda u} \frac{du}{u} + o\left(\frac{\sqrt{n}}{|x| \lg n}\right). \end{aligned} \quad (17)$$

Next suppose $|x| \leq \sqrt{n}/\lg n$ and effect a similar summation over $k \geq n$ of the right side of (14) (with k replacing n) and subtract the resulting sum from unity, and then, recalling (12) you find that

$$\begin{aligned} P_x[\sigma_{\{0\}} < n] &= 1 - \frac{2\kappa a^\dagger(x)}{\lg n} (1 + o(1)) + O\left(\frac{x^2}{n \lg n}\right) \\ &= \frac{0 \vee \lg(n/x^2)}{\lg n} + o(1). \end{aligned} \quad (18)$$

By a little reflection these two relations show (16) as desired. \square

REMARK 2. Formula (16) does not identify the leading order of $P_x[\sigma_{\{0\}} < n]$ even within $|x| = O(\sqrt{n})$: eg., if $|x| \sim \sqrt{n}/\lg n$, the error term reduces to $o(1)$ while the leading term is $O((\lg \lg n)/\lg n)$. However, if $E[X^2 \lg |X|] < \infty$, then $a(x) = \kappa^{-1} \lg \|x\| + c^* + o(1)$ as $|x| \rightarrow \infty$ for some constant c^* [9, Theorem 1] and from (18) we have

$$P_x[\sigma_{\{0\}} < n] = \frac{\lg(n/\tilde{x}^2)}{\lg n} + \frac{2\kappa c^*}{\lg n} (1 + o(1)) + O\left(\frac{x^2}{n \lg n}\right)$$

as $|x| \wedge n \rightarrow \infty$, which combined with (17) (valid for $|x| \geq \sqrt{n}/\lg n$) provides an explicit asymptotic form of the probability within the regime $|x| = O(\sqrt{n})$.

Let $(\eta_n)_{n=1}^\infty$ be a sequence of real i.i.d. random variables of mean zero with a finite and positive variance, and consider the random walk $Y_n = \eta_1 + \dots + \eta_n$ on \mathbb{R} . For convenience of later citation we record the following two lemmas. Although it may fall into a body of common knowledge, proofs are given for completeness.

Lemma 2.4. *Let Y_n be as above and σ_B^Y denote the first entrance time of the walk (Y_n) into B . Then, as $R \rightarrow \infty$*

$$\frac{1}{R} E[Y_{\sigma_{[R, \infty)}^Y} - R] \longrightarrow 0.$$

Proof. Let Z be the first strict ascending ladder variable in (Y_n) , i.e., $Z = Y_{\sigma_{(0, \infty)}^Y}$ and $V(dx)$ the renewal measure of the ascending ladder process: $V(dx) = \sum_{n=1}^\infty P[Z_n \in dx]$, where Z_n is the n -th record value of Y . Then

$$E[Y_{\sigma_{[R, \infty)}^Y} - R] = \int_{(0, R)} V(dy) \int_{[R-y, \infty)} z P[Z \in dz].$$

On knowing $EZ < \infty$ [1, Theorem 18.5.1] and $V((0, R]) \leq CR$, this is $o(R)$ as readily ascertained. \square

Lemma 2.5. *Let Y_n be as above and $\sigma^2 := EY_1^2$. Then there exists a universal constant $0 < c_0 < 1$ such that for $N > 0$ and for all sufficiently large R ,*

$$\sup_{x \in \mathbb{R}: |x| < R} P[|Y_k + x| < R \text{ for } k \leq N] \leq c_0^{-1 + \sigma^2 N/R^2}.$$

Proof. From central limit theorem it follows that

$$\limsup_{R \rightarrow \infty} \sup_{|x| < R} P[|Y_k + x| < R \text{ for } k \leq R^2/\sigma^2] \leq \lim_{R \rightarrow \infty} P[|Y_{\lfloor R^2/\sigma^2 \rfloor}| \leq R] =: c,$$

where $c = \int_{-1}^1 \frac{e^{-u^2/2} du}{\sqrt{2\pi}}$. Thus, taking any c_0 from $(c, 1)$ we find that if $n = \lfloor \sigma^2 N/R^2 \rfloor$,

$$P[|Y_k + x| < R \text{ for } k \leq N] \leq c_0^n \quad (|x| < R)$$

for all sufficiently large R . □

2.2 Overshoot estimates

Proposition 2. *Uniformly for $x \in U(R)$, as $R \rightarrow \infty$*

$$\frac{1}{R} E_x[|S_{\tau_{U(R)}}| - R] \rightarrow 0.$$

Proof. It suffices to show that for each $\varepsilon > 0$,

$$\frac{1}{R} E_x[|S_{\tau_{U(R)}}| - R; |S_{\tau_{U(R)}}| > (1 + \varepsilon)R] \rightarrow 0. \quad (19)$$

For the proof we divide the plane by several rays emanating from the origin to reduce the problem to a one-dimensional one. Taking a positive integer N so that

$$(1 + \varepsilon) \cos(\pi/N) \geq 1, \quad (20)$$

we put

$$W_1 = \{y = (y_1, y_2) \in \mathbb{Z}^2 : y_1 \geq (|y_2| \tan \frac{\pi}{N}) \vee R\},$$

the intersection of the half plane $y_1 \geq R$ and the infinite sector held by the two rays $y_1 = \pm y_2 \tan \frac{\pi}{N}, y_1 > 0$, and let W_k ($k = 2, \dots, N$) be the rotation of W_1 by $2k\pi/N$ counterclockwise. Then, by virtue of (20) the event $\{|S_{\tau_{U(R)}}| > (1 + \varepsilon)R\}$ occurs only if one of the N events

$$\Lambda_k := \{S_{\tau_{U(R)}} \in W_k\}, \quad k = 1, 2, \dots, N$$

occurs, hence the expectation to be estimated is dominated by $\sum E_x[|S_{\tau_{U(R)}}| - R; \Lambda_k]$, of which we consider the term with $k = 1$ for convenience of description. Let Y_n be the first component of S_n : $Y_n = \mathbf{e} \cdot S_n$. Noting that the occurrence of Λ_1 entails $Y_{\tau_{U(R)}} \geq R$ we see

$$E_x[|S_{\tau_{U(R)}}| - R; \Lambda_1] \leq 2E_x[Y_{\sigma_{[R, \infty)}^Y} - R; \tau_{U(R)} = \sigma_{[R, \infty)}^Y] = o(R).$$

Here the last equality follows from Lemma 2.4, for the expectation on the middle member is at most $E_0[Y_{\sigma_{[R', \infty)}^Y} - R']$ with $R' = R - \mathbf{e} \cdot x \leq 2R$. By the same argument we obtain $E_x[|S_{\tau_{U(R)}}| - R; \Lambda_k] = o(R)$ for each k . Thus the relation (19) is verified. □

By Markov's inequality there follows immediately from Proposition 2 the following

Corollary 2. *Uniformly for $x \in U(R)$ and for $R' > R$, as $R \rightarrow \infty$*

$$P_x[|S_{\tau_{U(R)}}| > R'] = o\left(\frac{R}{R' - R}\right). \quad (21)$$

Lemma 2.6. *Uniformly for $x \in U(R)$, as $R \rightarrow \infty$*

$$E_x\left[\lg(|S_{\tau_{U(R)}}|/R); \tau_{U(R)} < \sigma_A\right] = P_x[\tau_{U(\sqrt{R})} < \sigma_A] \times o(1) + o(R^{-1/2}). \quad (22)$$

Proof. Write $\Delta = \lg(|S_{\tau_{U(R)}}|/R)$. By the inequality $\lg(u/R) \leq (u - R)/R$ ($u > R$)

$$E_x[\Delta; \tau_{U(R)} < \sigma_A] \leq \frac{1}{R} E_x[|S_{\tau_{U(R)}}| - R; \tau_{U(R)} < \sigma_A].$$

The right side tending to zero according to Proposition 2, (22) holds if $\sqrt{R} \leq |x| < R$, since then the probability on the right side of it is bounded away from zero.

As for $x \in U(\sqrt{R})$ we decompose

$$E_x[\Delta; \tau_{U(R)} < \sigma_A] = E_x[\Delta; \tau_{U(\sqrt{R})} = \tau_{U(R)} < \sigma_A] + E_x[\Delta; \tau_{U(\sqrt{R})} < \tau_{U(R)} < \sigma_A].$$

The second term on the right side is written as

$$E_x[E_{S_{\tau_{U(\sqrt{R})}}}[\Delta; \tau_{U(R)} < \sigma_A]; \tau_{U(\sqrt{R})} < \tau_{U(R)} \wedge \sigma_A],$$

which, owing to what we have observed above, is evaluated to be $P_x[\tau_{U(\sqrt{R})} < \sigma_A] \times o(1)$. On the other hand the first term is dominated by

$$\begin{aligned} E_x[\Delta; S_{\tau_{U(\sqrt{R})}} \geq R] &\leq 2P_x[R \leq S_{\tau_{U(\sqrt{R})}} < R^2] + E_x[\lg |S_{\tau_{U(\sqrt{R})}}|; |S_{\tau_{U(\sqrt{R})}}| \geq R^2] \\ &\leq o(R^{-1/2}) + R^{-1} E_x[|S_{\tau_{U(\sqrt{R})}}|^{1/2} \lg |S_{\tau_{U(\sqrt{R})}}|] \\ &= o(R^{-1/2}), \end{aligned}$$

where we have applied Corollary 2 and Markov's inequality for the second inequality and Proposition 2 for the last relation. The proof of Lemma 2.6 is complete. \square

Lemma 2.7. *Uniformly for $x \in U(R)$, as $R \rightarrow \infty$*

$$E_x[|\kappa a(S_{\tau_{U(R)}}) - \lg R|; \tau_{U(R)} < \sigma_A] = (1 \vee \lg |x|) \times o(1).$$

If $E[X^2 \lg |X|] < \infty$, then the right side can be replaced by $P_x[\tau_{U(\sqrt{R})} < \sigma_A] \times O(1) + o(R^{-1/2})$.

Proof. We may and do suppose $0 \in A$, so that a is harmonic on $\mathbb{Z}^2 \setminus A$. By the optional stopping theorem applied to the positive martingale $a(S_{n \wedge \sigma(A)})$ it follows that

$$0 \leq E_x[a(S_{\tau_{U(R)}}); \tau_{U(R)} < \sigma_A] \leq a^\dagger(x)$$

for all x . (Note the case $x \in A$ is reduced to the case $x \notin A$.) In the decomposition

$$\kappa a(S_{\tau_{U(R)}}) - \lg R = \kappa a(S_{\tau_{U(R)}}) - \lg |S_{\tau_{U(R)}}| + \lg(|S_{\tau_{U(R)}}|/R),$$

the contribution of the first two terms on the right side is therefore disposed of as follows:

$$\begin{aligned} & E_x \left[\left| \kappa a(S_{\tau_{U(R)}}) - \lg |S_{\tau_{U(R)}}| \right| ; \tau_{U(R)} < \sigma_A \right] \\ &= \begin{cases} a^\dagger(x) \times o(1) & \text{in general,} \\ P_x[\tau_{U(R)} < \sigma_A] \times O(1) & \text{if } E[X^2 \lg |X|] < \infty, \end{cases} \end{aligned}$$

where we have also applied the fact that $\kappa a(z) - \lg |z|$ is $o(a(z))$ ($|z| \rightarrow \infty$) in general and bounded if $E[X^2 \lg |X|] < \infty$ ([9, Theorem1]), whereas the required bound concerning the third term is provided by Lemma 2.6. \square

In the next lemma we include the case when the existence of higher moments are assumed though not applied in this article.

Lemma 2.8. *Suppose $E[|X|^{2+\delta}] < \infty$ for a constant $\delta \geq 0$. Then, uniformly for $R > 1$ and $x \in U(R)$,*

$$P_x[S_{\tau_{U(R)}} = y, \tau_{U(R)} < \sigma_A] = a^\dagger(x) \times o((|y| - R)^{-(2+\delta)}) \quad (|y| - R \rightarrow \infty),$$

and

$$P_x[|S_{\tau_{U(R)}}| > R + h, \tau_{U(R)} < \sigma_A] = a^\dagger(x) R^2 \times o(h^{-(2+\delta)}) \quad (h \rightarrow \infty).$$

Proof. Take R such that $A \subset U(R)$ and put $D(R) = A \cup (\mathbb{Z}^2 \setminus U(R))$. Then for $y \notin U(R)$,

$$P_x[S_{\tau_{U(R)}} = y, \tau_{U(R)} < \sigma_A] = \sum_{z \in U(R) \setminus A} g_{D(R)}(x, z) p(y - z).$$

Suppose $0 \in A$ for simplicity. Then, $g_{D(R)}(x, z) \leq g_{\{0\}}(x, z) \leq g_{\{0\}}(x, x) \leq C a^\dagger(x)$. Hence

$$P_x[S_{\tau_{U(R)}} = y, \tau_{U(R)} < \sigma_A] \leq C a^\dagger(x) P[|y - X| < R] \leq C a^\dagger(x) P[|X| > |y| - R]$$

($y \notin U(R)$) and an application of Markov's inequality yields the first relation of the lemma. To verify the second relation we use the first inequality above and observe

$$\sum_{|y| > R+h} P[|y - X| < R] = \sum_{|z| < R} \sum_{|y| > R+h} p(y - z) = 4R^2 P[|X| > h],$$

which implies the required bound in view of Markov's inequality. \square

2.3 The function u_A and proof of Proposition 1

We define a function $u_A(x)$ by

$$u_A(x) = g_A(x, y) + a(x - y) - E_x[a(S_{\sigma(A)} - y)], \quad y \in \mathbb{Z}^2. \quad (23)$$

Lemma 2.9. *The right side of (23) is independent of $y \in \mathbb{Z}^2$ for all $x \in \mathbb{Z}^2$. In particular for any $\xi_0 \in A$*

$$u_A(x) = \delta(x, \xi_0) + a(x - \xi_0) - E_x[a(S_{\sigma(A)} - \xi_0)]. \quad (24)$$

Proof. With $x \in \mathbb{Z}^2$ fixed put $f(y) := g_A(x, y) + a(x - y) - E_x[a(S_{\sigma(A)} - y)]$. We are to prove that f is dual-harmonic on \mathbb{Z}^2 , namely $f(y) = \sum_z p(y - z)f(z)$ for all $y \in \mathbb{Z}^2$, hence must be constant, for f is bounded (cf. [3, Proposition 6.3]). Remember that for all $x, y \in \mathbb{Z}^2$,

$$g_A(x, y) = \sum_{n=0}^{\infty} p_A^n(x, y), \quad p_A^0(x, y) = \delta_{x, y};$$

in particular $g_A(x, y) = \delta_{x, y}$ for $y \in A$. We then observe that if \hat{P} denotes the dual transition operator, i.e., $\hat{P}f(y) = \sum_z p(y - z)f(z)$, then for all $x \in \mathbb{Z}^2$,

$$\hat{P}a(x - \cdot)(y) = a(x - y) + \delta_{x, y}$$

and

$$\hat{P}\{E_x[a(S_{\sigma(A)} - \cdot)]\}(y) = \begin{cases} E_x[a(S_{\sigma(A)} - y)] & (y \notin A), \\ E_x[a(S_{\sigma(A)} - y)] + P_x[S_{\sigma_A} = y] & (y \in A), \end{cases}$$

while

$$\hat{P}g_A(x, \cdot)(y) = \begin{cases} g_A(x, y) - \delta_{x, y} & (y \notin A), \\ P_x[S_{\sigma_A} = y] & (y \in A). \end{cases}$$

From these there readily follows the identity $\hat{P}f = f$ as required. \square

REMARK 3. The assertion of Lemma 2.9 as well as the proof given above is valid for all recurrent walks. For x restricted on A , (23) reduces to the dual of the formula of Proposition 30.1 in [7], where the dual of $u_A(x)$, which therein is denoted by $\mu_A(x)$, is defined as the limit of $\sum_{z \in \mathbb{Z}^2} p^n(z - y)P_z[S_{\sigma(A)} = x]$ as $n \rightarrow \infty$. On the other hand for $x \notin A$ an equivalent to (23) is verified in [5, Proposition 4.6.3] in a different way when X is of finite range and symmetric. Formula (23) is suggested by a similar one used by Hunt [4] to define the (classical) Green function in a plane region with pole at infinity.

By virtue of (13) passing to the limit in (23) yields

$$u_A(x) = \lim_{|y| \rightarrow \infty} g_A(x, y).$$

Thus the present definition of u_A agrees with the one given in (4). By (12) and (24) we also obtain that $\kappa u_A(x) \sim \lg |x|$ ($|x| \rightarrow \infty$) as stated in (5).

Now we prove Proposition 1, which we state again

Proposition 1. *Uniformly for $x \in U(R)$, as $R \rightarrow \infty$*

$$P_x[\tau_{U(R)} < \sigma_A] = \frac{\kappa u_A(x)}{\lg R}(1 + o(1));$$

and if $E[X^2 \lg |X|] < \infty$, then the error term $o(1)$ may be replaced by $O(1/\lg R)$.

Proof. It suffices to prove the postulated formula for $x \notin A$, the case $x \in A$ being reduced to it by virtue of (7). Suppose $x \notin A$ and let $\xi \in A$ and $D(R) = A \cup (\mathbb{Z}^2 \setminus U(R))$ (for R large enough). We adapt the proof in [5] (Proposition 6.4.7) to the present situation of unbounded X (cf. also [8] (Theorem 11.2.14)). By optional sampling theorem the process $M_n := a(S_{n \wedge \sigma_{D(R)}} - \xi)$ is a martingale under P_x . It is easy to see that M_n is L^2 bounded and we apply the martingale convergence theorem to have

$$a(x - \xi) = E_x[a(S_{\sigma_{D(R)}} - \xi)].$$

Breaking this expectation according as the first visit to $D(R)$ occurs on A or on $\mathbb{Z}^2 \setminus U(R)$ yields

$$\begin{aligned} a(x - \xi) - E_x[a(S_{\sigma_A} - \xi)] &= E_x[a(S_{\tau_{U(R)}} - \xi); \tau_{U(R)} < \sigma_A] \\ &\quad - E_x[a(S_{\sigma_A} - \xi); \tau_{U(R)} < \sigma_A]. \end{aligned}$$

Recall that by (24) the left side equals $u_A(x)$ ($x \notin A$). Then, putting

$$c(x, R) = E_x[a(S_{\sigma_A} - \xi) \mid \tau_{U(R)} < \sigma_A]$$

and

$$\theta(x, R) = E_x[a(S_{\tau_{U(R)}} - \xi) - \kappa^{-1} \lg R; \tau_{U(R)} < \sigma_A], \quad (25)$$

we rewrite the identity above in the form

$$u_A(x) = \theta(x, R) + (\kappa^{-1} \lg R - c(x, R))P_x[\tau_{U(R)} < \sigma_A]. \quad (26)$$

Since $c(x, R)$ is bounded and in view of Lemma 2.7 and (5) $\theta(x, R) = u_A(x) \times o(1)$ as $R \rightarrow \infty$ uniformly for $x \in U(R) \setminus A$, on dividing by $\kappa^{-1} \lg R$ we obtain the formula of the proposition. If $E[X^2 \lg |X|] < \infty$, by the second assertion of Lemma 2.7 and what is just proved we have $\theta(x, R) = P_x[\tau_{U(R)} < \sigma_A] \times O(1)$, and substitution of it into (26) leads to the desired estimate of the error term. \square

3 Proof of Theorem 1

Throughout this section let $d = 2$.

Proposition 3. *As $|x| \wedge |y| \wedge n \rightarrow \infty$, under the condition $\liminf [\lg(x^2 \wedge y^2)] / \lg n \geq 1$,*

$$p_A^n(x, y) = p^n(y - x) + o\left(\frac{1}{n \vee x^2 \vee y^2}\right).$$

Proof. Let $|x| \leq |y|$ and $p^n(y - x) > 0$. We make decomposition

$$p^n(y - x) - p_A^n(x, y) = \sum_{k=1}^n \sum_{\xi \in A} P_x[\sigma_A = k, S_k = \xi] p^{n-k}(y - \xi), \quad (27)$$

which entails

$$p^n(y - x) - p_A^n(x, y) \leq \sum_{k=1}^n \sum_{\xi \in A} P_x[\sigma_{\{\xi\}} = k] p^{n-k}(y - \xi). \quad (28)$$

Let $I_{[a,b]}$ denote the latter double sum restricted to the interval $[a, b]$, $0 \leq a < b \leq n$.

By Lemma 2.2 (local limit theorem) it follows that for $\xi \in A, k < n/2$,

$$p^{n-k}(y - \xi) \leq p^n(y) + o(1/(n \vee y^2)) = O(1/(n \vee y^2)), \quad (29)$$

hence

$$I_{[0, n/2]} \leq \frac{C}{n \vee y^2} \sum_{\xi \in A} P_x[\sigma_{\{\xi\}} \leq n/2].$$

Applying Lemma 2.3 to $P_x[\sigma_{\{\xi\}} \leq n/2]$ we deduce that

$$I_{[0, n/2]} \leq \frac{C'}{n \vee y^2} \left(\frac{1 \vee \lg(n/x^2)}{\lg n} + o\left(1 \wedge \frac{\sqrt{n}}{|x| \lg n}\right) \right). \quad (30)$$

For the other part $I_{[n/2, n]}$ we apply (15) and observe

$$I_{[n/2, n]} \leq \frac{C}{n \lg n} \sum_{\xi \in A} \sum_{n/2 < k \leq n} p^{n-k}(y - \xi). \quad (31)$$

The inner sum $J_n := \sum_{n/2 < k \leq n} p^{n-k}(y - \xi)$ is estimated somewhat differently depending on whether $y^2 > n/2$ or not. If $y^2 \leq n/2$, then, on splitting the sum at $j := n - k = \lfloor y^2 \rfloor$ and using Lemma 2.2,

$$J_n = \sum_{j \leq y^2} + \sum_{y^2 < j \leq n/2} \leq \int_0^{y^2} \left(p_{\sigma^2 t}^{(2)}(\|y\|) + o\left(\frac{1}{y^2}\right) \right) dt + \sum_{y^2 \leq j \leq n/2} \frac{1}{j} = O\left(\lg(n/y^2)\right),$$

which we may write as

$$J_n \leq C' n p^n(y) \lg(n/y^2).$$

Similarly, if $y^2 > n/2$, then

$$J_n = \int_0^{n/2} p_{\sigma^2 t}^{(2)}(\|y\|) dt + \sum_{1 \leq j < n/2} o\left(\frac{1}{j \vee y^2}\right) \leq C \frac{n}{y^2} e^{-\|y\|^2/2\sigma^2 n} + o\left(\frac{1}{y^2}\right) \times n. \quad (32)$$

In view of (31) these two bounds of J_n show

$$I_{[n/2, n]} \leq \frac{C' \sharp A}{n \lg n} J_n = p^n(y) \times O\left(\frac{1 \vee \lg(n/y^2)}{\lg n}\right) + \frac{1}{\lg n} \times o\left(\frac{1}{n \vee y^2}\right). \quad (33)$$

By the ‘liminf’ condition we have $1 \vee \lg(n/y^2) = o(\lg n)$, hence the right side above is $o(1/(n \vee y^2))$; and similarly for the right side of (30). This finishes the proof. \square

REMARK 4. The estimate of Proposition 3 may be improved under a higher moment condition. Suppose $E|X|^{2+\delta} < \infty$ for a constant $\delta \geq 0$ and $|x| \leq |y|$. Then, for any $M > 1$, as $|x| \wedge n \rightarrow \infty$ under the constraint $(1 \vee |x|)|y| < Mn$

$$p_A^n(x, y) = p^n(y - x) \left[1 + O\left(\frac{1 \vee \lg(n/x^2)}{\lg n}\right) \right] + o\left(\frac{1}{(n \vee y^2)^{1+\delta/2}}\right). \quad (34)$$

The proof is carried out by examining the preceding one. Owing to Lemma 2.2, $o(1/(n \vee y^2))$ may be replaced by $o((n \vee y^2)^{-1-\delta/2})$ in (29). Taking Remark 2 (given to Lemma 2.3) into account, this leads to, instead of (30),

$$I_{[0, n/2]} \leq \left[C p^n(y) + o\left(\frac{1}{(n \vee y^2)^{1+\delta/2}}\right) \right] \frac{1 \vee \lg(n/x^2)}{\lg n}. \quad (35)$$

As for $I_{[n/2, n]}$, using (31) as before we infer that on the right side of (33) $o(1/(n \vee y^2))$ can be replaced by $o(1/(n \vee y^2)^{1+\delta/2})$. Combined with (35) this shows (34), since our supposition entails $|y - x|^2 - y^2 = O(n)$, so that $p^n(y)$ can be replaced by $p^n(y - x)$. The details are omitted.

Proposition 4. As $n \wedge |y| \rightarrow \infty$ under the constraint $\liminf (\lg y^2)/\lg n \geq 1$,

$$p_A^n(x, y) = \frac{2\kappa u_A(x)}{\lg n} \left[p^n(y - x) + o\left(\frac{1}{n \vee y^2}\right) \right] \quad (36)$$

uniformly for $x \in U(\sqrt{n})$.

Proof. Suppose $p^n(y - x) > 0$. Define $R = R_n$ and $N = N_n$ by

$$R = \frac{\sqrt{n}}{(\lg n)^2} \quad \text{and} \quad N = \lfloor (R \lg n)^2 \rfloor,$$

so that

$$(a) \quad \frac{N}{n} = O\left(\frac{1}{(\lg n)^2}\right) \quad \text{and} \quad (b) \quad \frac{N+1}{R^2} \geq (\lg n)^2. \quad (37)$$

The case $R \leq |x| < \sqrt{n}$ being covered by Proposition 3 because of (5), in the sequel we suppose that $|x| < R$.

Define $\varepsilon(n, x, y)$ via

$$\begin{aligned} p_A^n(x, y) &= \sum_{k=1}^{N-1} \sum_{z \notin U(R)} P_x[\tau_{U(R)} = k < \sigma_A, S_{\tau_{U(R)}} = z] p_A^{n-k}(z, y) \\ &\quad + \varepsilon(n, x, y). \end{aligned} \quad (38)$$

By Lemma 2.5 we have

$$P_x[\tau_{U(R)} \wedge \sigma_A \geq N] \leq c e^{-\lambda N/R^2} \quad (39)$$

with a constant $\lambda > 0$ (depending only on the walk) and, noting that for $z \in U(R)$, $p_A^{n-N}(z, y) \leq p^{n-N}(y - z) \leq c p^n(y - x) + o(n^{-1} \wedge y^{-2})$, we observe

$$\begin{aligned} \varepsilon(n, x, y) &= \sum_{z \in U(R) \setminus A} P_x[\tau_{U(R)} \wedge \sigma_A \geq N, S_N = z] p_A^{n-N}(z, y) \\ &\leq c' e^{-\lambda N/R^2} \left[p^n(y - x) + o(n^{-1} \wedge y^{-2}) \right], \end{aligned} \quad (40)$$

hence $\varepsilon(n, x, y)$ is absorbed into the error term in (36) because of (37:(b)).

To evaluate the double sum in (38) we take

$$r = r_{y,n} = (\sqrt{n} \vee |y|)/\sqrt{\lg n},$$

so that

$$\frac{R^2}{r^2} = \frac{n}{(n \vee y^2)(\lg n)^3}, \quad (41)$$

and first dispose of the part with the inner sum restricted to the outside of $U(r)$. Since for $k \leq N$,

$$p_A^{n-k}(z, y) \leq C/n,$$

we see that this part is

$$\begin{aligned} &\sum_{k=1}^{N-1} \sum_{z \notin U(r)} P_x[\tau_{U(R)} = k < \sigma_A, S_{\tau_{U(R)}} = z] p_A^{n-k}(z, y) \\ &\leq \frac{C}{n} P_x[\tau_{U(R)} < \sigma_A, S_{\tau_{U(R)}} \notin U(r)], \end{aligned} \quad (42)$$

hence negligible, for by Lemma 2.8 and (41) the probability on the right side is dominated by a constant multiple of $a^\dagger(x)R^2/r^2 = O\left(n/[(n \vee y^2)(\lg n)^2]\right)$.

For $k < N$, $z \in U(r) \setminus U(R)$ with $p^{n-k}(y-z) > 0$, by Proposition 3 we have

$$p_A^{n-k}(z, y) = p^{n-k}(y-z) + o(1/(n \vee y^2));$$

also we have

$$p^{n-k}(y-z) = p^n(y-x) + o(1/(n \vee y^2)),$$

where we have applied $|z| = o(\sqrt{n} \vee |y|)$ and $|x| = O(\sqrt{n})$. Thus the contribution to the double sum to be evaluated is written as

$$P_x[\tau_{U(R)} < N \wedge \sigma_A] \times [p^n(y-x) + o(1/(n \vee y^2))],$$

while, owing to Proposition 1 and (39),

$$\begin{aligned} P_x[\tau_{U(R)} < N \wedge \sigma_A] &= P_x[\tau_{U(R)} < \sigma_A] - P_x[N \leq \tau_{U(R)} < \sigma_A] \\ &= \frac{\kappa u_A(x)}{\lg R}(1 + o(1)). \end{aligned} \tag{43}$$

Now, combining this with what we have observed by (42), we conclude that the double sum in (38) is written as

$$\frac{\kappa u_A(x)}{\lg R} \left[p^n(y-x) + o\left(\frac{1}{n \vee y^2}\right) \right].$$

Since $\lg R = \frac{1}{2}(\lg n)(1 + o(1))$ as noted previously, this verifies the lemma. \square

Proof of Theorem 1. The relation to verify is

$$p_A^n(x, y) = \frac{4\kappa^2 u_A(x)u_{-A}(-y)}{(\lg n)^2} p^n(y-x)(1 + o(1)).$$

In view of Propositions 3 and 4 we may suppose $|x| \vee |y| \leq \sqrt{n}/(\lg n)^2$. Consider the double sum in (38) (with the same R and N as therein), of which we have observed that the contribution to it from $|z| > \sqrt{n/\lg n}$ may be neglected, while by Proposition 4 we have for $R \leq |z| \leq \sqrt{n/\lg n}$

$$p_A^{n-k}(z, y) = p_{-A}^{n-k}(-y, -z) = \frac{2\kappa u_{-A}(-y)}{\lg n} p^n(y-z)(1 + o(1)).$$

Since $p^n(y-z) = p^n(y-x)(1 + o(1))$ if both sides are positive, we may write (38) as

$$p_A^n(x, y) = P_x[\tau_{U(R)} < N \wedge \sigma_A] \frac{2\kappa u_{-A}(-y)}{\lg n} p^n(y-x)(1 + o(1)) + \varepsilon(n, x, y).$$

The evaluation of the term $\varepsilon(n, x, y)$ given in (40) and that of $P_x[\tau_{U(R)} < N \wedge \sigma_A]$ in (43) are both available. Thus we conclude the required relation of Theorem 1. \square

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